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A Behavioral Input-Output Parametrization of Control Policies with Suboptimality Guarantees

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***Abstract*— Recent work in data-driven control has revived behavioral theory to perform a variety of complex control tasks, by directly plugging libraries of past input-output trajectories into optimal control problems. Despite recent advances, a key aspect remains unclear: how and to what extent do noise-corrupted data impact control performance? In this work, we provide a quantitative answer to this question based on the model-mismatch level incurred during a preliminary system identification phase. We formulate a Behavioral version of the Input-Output Parametrization (BIOP) for the optimal predictive control of unknown systems using output-feedback dynamic control policies. The main advantages of the proposed framework are that 1) the state-space parameters and the initial state need not be specified for controller synthesis, 2) it can be used in combination with state-of-the-art impulse response estimators, and 3) it allows to recover suboptimality results on learning the Linear Quadratic Gaussian (LQG) controller, therefore revealing how the model-mismatch level may affect the performance. Specifically, it is shown that the performance degrades linearly with the model-mismatch incurred by either classical or behavioral-based system identification.**

I. Introduction

Several safety-critical engineering systems that play a crucial role in our modern society are becoming too complex to be accurately modeled through white-box models [1]. As a consequence, most modern control perspectives envision unknown black-box systems for which an optimal behavior must be attained by solely relying on a collection of system's output trajectories in response to different inputs recorded offline.

Widely speaking, we can design optimal controllers from data according to two paradigms. The first category contains *model-based* methods, where historical input-output trajectories are exploited to approximate the system parameters, and a suitable controller is computed for this estimated model. The second category contains *model-free* methods, where one aims to learn the best control policy directly by observing historical trajectories, without explicitly reconstructing an internal representation of the dynamical system. Both approaches possess their own potential and limitations; among numerous recent surveys, we refer to [2].

Given the intricacy of establishing rigorous suboptimality and sample-complexity bounds, most recent model-based and model-free approaches have focused on basic Linear Quadratic Regulator (LQR) and Linear Quadratic Gaussian (LQG) control problems as suitable benchmarks to establish how machine learning can be interfaced to the continuous action spaces typical of control [3]–[6].

A promising data-driven approach that aims at bypassing a parametric description of the system dynamics, while still being conceptually simple to implement for the users, hinges on the *behavioral framework* [7]. This approach has gained renewed interest with the data-driven methods of [8], which established that constrained output reference tracking can be effectively tackled in a Model-Predictive-Control (MPC) fashion by plugging adequately generated historical data into a convex optimization problem. In parallel, [9] introduced data-driven formulations for some controller design tasks. These works inspired several extensions including closed-loop control with stability guarantees [10], maximum-likelihood identification for control [11], [12], and nonlinear variants [13].

In practice, however, historical data are corrupted by noise and the quality and coherency of the achieved solutions may be compromised. While several approaches have recently been proposed, e.g. [12], [14], a complete quantitative analysis for the noisy case is still unavailable. Recently, [15] has derived suboptimality and sample-complexity bounds through a data-driven formulation of the System Level Synthesis (SLS) approach. However, a limiting assumption in [15] is that the internal system states can be measured directly, which is infeasible for several large-scale systems [16].

Our main contribution is to propose a behavioral optimal control framework for partially observed systems. Specifically, we leverage recent Input-Output Parametrization (IOP) tools [17] for optimal output-feedback controller design and set up a data-driven formulation built upon behavioral theory; we denote the resulting framework as Behavioral IOP (BIOP). The advantages of the proposed BIOP are threefold. First, it solely relies on libraries of past input-output trajectories, therefore enabling optimal controller synthesis without specifying the system's state-space parameters and the value of the state at the initial time. Second, the system impulse response is replaced by a suitable linear combination of historical noisy input-output trajectories, which may encompass, for instance, standard least-squares solutions [18], and the recently proposed signal matrix models (SMM) [11], [12]. Third, our framework allows one to quantify the incurred suboptimality as a function of the model-mismatch level arising from a preliminary identification phase based on noisy data. This is achieved by first establishing a tractable

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method to synthesize robust BIOP controllers and then adapting recent results from [5]. As a further contribution, we include the effect of a non-zero noisy initial condition in the analysis.

We formulate the standard LQG control problem in Section II and we present its model-based solution based on the IOP [17] in Section III; this form enables the analysis later. Section IV derives the BIOP, a data-driven version of the IOP valid when the data are noiseless. Section V establishes a tractable robust version of the BIOP which can be used when the data are noisy. Section VI formally quantifies the suboptimality incurred by the solution of the robust BIOP. We conclude the paper in Section VII. The accompanying Arxiv report [19] provides proofs in the appendices, as well as numerical experiments and a few additional remarks.

A. Notation

We use $\mathbb{R}$ and $\mathbb{N}$ to denote real numbers and non-negative integers, respectively. We use I_n to denote the identity matrix of size $n \times n$ and $0_{m\times n}$ to denote the zero matrix of size $m \times n$. We write $M = \text{blkdg}(M_1, \ldots, M_N)$ to denote a block-diagonal matrix with $M_1, \ldots, M_N \in \mathbb{R}^{m\times n}$ on its diagonal block entries, and for $\mathbf{M} = \begin{bmatrix} M_1^\mathsf{T} & \cdots & M_N^\mathsf{T} \end{bmatrix}^\mathsf{T}$ we define the block-Toeplitz matrix

$$\text{Toep}_{m\times n}(\mathbf{M}) = \begin{bmatrix} M_1 & 0_{m\times n} & \cdots & 0_{m\times n} \\ M_2 & M_1 & \cdots & 0_{m\times n} \\ \vdots & \vdots & \ddots & \vdots \\ M_N & M_{N-1} & \cdots & M_1 \end{bmatrix}.$$

More concisely, we will write $\text{Toep}(\cdot)$ when the dimensions of the blocks are clear from the context. The Kronecker product between $M \in \mathbb{R}^{m\times n}$ and $P \in \mathbb{R}^{p\times q}$ is denoted as $M \otimes P \in \mathbb{R}^{mp\times nq}$. Given $K \in \mathbb{R}^{m\times n}$, $\text{vec}(K) \in \mathbb{R}^{mn}$ is a column vector that stacks the columns of K. The Euclidean norm of a vector $v \in \mathbb{R}^n$ is denoted by $\|v\|_2^2 = v^\mathsf{T} v$ and the induced two-norm of a matrix $M \in \mathbb{R}^{m\times n}$ is defined as $\sup_{\|x\|_2=1} \|Mx\|_2$. The Frobenius norm of a matrix $M \in \mathbb{R}^{m\times n}$ is denoted by $\|M\|_F = \sqrt{\text{Trace}(M^\mathsf{T} M)}$. For a symmetric matrix M, we write $M \succ 0$ (resp. $M \succeq 0$) if and only if it is positive definite (resp. positive semidefinite). We say that $x \sim \mathcal{N}(\mu, \Sigma)$ if the random variable $x \in \mathbb{R}^n$ is distributed according to a normal distribution with mean $\mu \in \mathbb{R}^n$ and covariance matrix $\Sigma \succeq 0$ with $\Sigma \in \mathbb{R}^{n\times n}$.

A *finite-horizon* trajectory of length T is a sequence $\omega(0), \omega(1), \cdots, \omega(T-1)$ with $\omega(t) \in \mathbb{R}^n$ for every $t = 0, 1, \ldots, T-1$, which can be compactly written as

$$\boldsymbol{\omega}_{[0,T-1]} = \begin{bmatrix} \omega^\mathsf{T}(0) & \omega^\mathsf{T}(1) & \ldots & \omega^\mathsf{T}(T-1) \end{bmatrix}^\mathsf{T} \in \mathbb{R}^{nT}.$$

When the value of T is clear from the context, we will omit the subscript $[0, T-1]$. For a finite-horizon trajectory $\boldsymbol{\omega}_{[0,T-1]}$ we also define the Hankel matrix of depth L as

$$\mathcal{H}_L(\boldsymbol{\omega}_{[0,T-1]}) = \begin{bmatrix} \omega(0) & \omega(1) & \cdots & \omega(T-L) \\ \omega(1) & \omega(2) & \cdots & \omega(T-L+1) \\ \vdots & \vdots & \ddots & \vdots \\ \omega(L-1) & \omega(L) & \cdots & \omega(T-1) \end{bmatrix}.$$

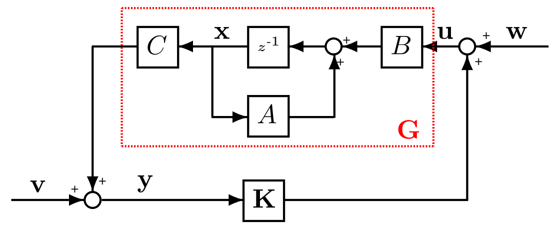

Fig. 1: Interconnection of the plant $\mathbf{G}$ and the controller $\mathbf{K}$, where z^{-1} denotes the standard time-shift operator.

II. Problem Statement

We consider a linear system with output observations, whose state-space representation is given by

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + v(t), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state of the system and $x(0) = x_0$ for a predefined $x_0 \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the observed output, and $v(t) \in \mathbb{R}^p$ denotes Gaussian measurement noise $v(t) \sim \mathcal{N}(0, \Sigma_v)$, with $\Sigma_v \succ 0$. The system is controlled through a time-varying, dynamic linear control policy of the form

$$u(t) = \sum_{k=0}^{t} K_{t,k} y(k) + w(t), \quad (2)$$

where $w(t) \in \mathbb{R}^m$ denotes Gaussian noise on the input $w(t) \sim \mathcal{N}(0, \Sigma_w)$ with $\Sigma_w \succeq 0$. Similar to standard LQG, our control goal is to synthesize a feedback control policy that minimizes the expected value with respect to the disturbances of a quadratic objective defined over future input-output trajectories for a horizon $N \in \mathbb{N}$:

$$J^2 := \mathbb{E}_{w,v}\left[\sum_{t=0}^{N-1} \left(y(t)^\mathsf{T} L_t y(t) + u(t)^\mathsf{T} R_t u(t)\right)\right], \quad (3)$$

where $L_t \succeq 0$, $R_t \succ 0$ for every $t = 0, \cdots, N-1$. We note that, with Gaussian noise, dynamic linear policies are optimal for the cost defined in (3).

III. Strongly convex design through the IOP

By leveraging tools offered by the framework of the IOP [17], we formulate a strongly convex program that computes the optimal feedback control policy by finding the optimal input-output closed-loop responses. The state-space equations (1) provide the following relations between trajectories

$$\mathbf{x}_{[0,N-1]} = \mathbf{P}_A(:,0)x(0) + \mathbf{P}_B \mathbf{u}_{[0,N-1]}, \quad (4)$$
$$\mathbf{y}_{[0,N-1]} = \mathbf{C}\mathbf{x}_{[0,N-1]} + \mathbf{v}_{[0,N-1]}, \quad (5)$$

where $\mathbf{P}_A(:,0)$ denotes the first block-column of $\mathbf{P}_A$ and

$$\mathbf{P}_A = (I - \mathbf{Z}\mathbf{A})^{-1}, \quad \mathbf{P}_B = (I - \mathbf{Z}\mathbf{A})^{-1}\mathbf{Z}\mathbf{B},$$

where $\mathbf{A} = I_N \otimes A$, $\mathbf{B} = I_N \otimes B$, $\mathbf{C} = I_N \otimes C$ and $\mathbf{Z}$ is the operator shifting all matrix blocks down by one position. We note that $\mathbf{CP}_B$ is a Toeplitz matrix with blocks in the form CA^iB. From now on, we equivalently denote $\mathbf{G} = \mathbf{CP}_B$ to highlight that $\mathbf{G}$ is a block-Toeplitz matrix containing the first N components of the impulse response of the plant $\mathbf{G}(z) = C(zI - A)^{-1}B$ reported in Figure 1. Second, with similar reasoning, the matrix $\mathbf{CP}_A(:,0)$ contains the observability terms CA^i for $i = 0, \ldots, N-1$. The control policy can be rewritten as:

$$\mathbf{u}_{[0,N-1]} = \mathbf{K}\mathbf{y}_{[0,N-1]} + \mathbf{w}_{[0,N-1]}\,, \tag{6}$$

where $\mathbf{K}$ has a causal sparsity pattern:

$$\mathbf{K} = \begin{bmatrix} K_{0,0} & 0_{m\times p} & \cdots & 0_{m\times p} \\ K_{1,0} & K_{1,1} & \ddots & 0_{m\times p} \\ \vdots & \vdots & \ddots & \vdots \\ K_{N-1,0} & K_{N-1,1} & \cdots & K_{N-1,N-1} \end{bmatrix}. \tag{7}$$

The input (6) can be thought of as a dynamic controller in convolutional form with initial state x_0. By plugging the controller (6) into (4)-(5), it is easy to derive the relationships

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} \begin{bmatrix} \mathbf{v} + \mathbf{CP}_A(:,0)x(0) \\ \mathbf{w} \end{bmatrix}, \tag{8}$$

where

$$\begin{bmatrix} \mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} = \begin{bmatrix} (I - \mathbf{GK})^{-1} & (I - \mathbf{GK})^{-1}\mathbf{G} \\ \mathbf{K}(I - \mathbf{GK})^{-1} & (I - \mathbf{KG})^{-1} \end{bmatrix}. \tag{9}$$

The parameters $(\mathbf{\Phi}_{yy}, \mathbf{\Phi}_{yu}, \mathbf{\Phi}_{uy}, \mathbf{\Phi}_{uu})$ represent the four closed-loop responses defining the relationship between disturbances and input-output signals. The main concept behind the IOP in [17] is that linear output-feedback control policies $\mathbf{K}$ can be expressed in terms of corresponding closed-loop responses that lie in an affine subspace, hence enabling a convex formulation of the objective $J(\mathbf{G}, \mathbf{K})$ given in (3) as a function of the closed-loop responses. The IOP serves well our purposes in a data-driven output-feedback setup, as it offers a controller parametrization that is directly defined through the impulse response parameters $\mathbf{G}$, without requiring a state-space representation. We adapt the following result from [17] to the finite horizon case. A proof is reported in the Appendix of [19].

Proposition 1: Consider the LTI system (1) evolving under the control policy (6) within a finite horizon of length $N \in \mathbb{N}$. Then:

1) For any controller $\mathbf{K}$ there exist four matrices $(\mathbf{\Phi}_{yy}, \mathbf{\Phi}_{yu}, \mathbf{\Phi}_{uy}, \mathbf{\Phi}_{uu})$ such that $\mathbf{K} = \mathbf{\Phi}_{uy}\mathbf{\Phi}_{yy}^{-1}$ and

$$\text{IOP}(\mathbf{\Phi}, \mathbf{G}) = 0\,,$$

where we define $\text{IOP}(\mathbf{\Phi}, \mathbf{G}) = 0$ as

$$\begin{bmatrix} I & -\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}, \tag{10}$$

$$\begin{bmatrix} \mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} \begin{bmatrix} -\mathbf{G} \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{11}$$

$$\mathbf{\Phi}_{yy}, \mathbf{\Phi}_{uy}, \mathbf{\Phi}_{yu}, \mathbf{\Phi}_{uu} \text{ have causal sparsities } ^1. \tag{12}$$

2) For any four matrices $(\mathbf{\Phi}_{yy}, \mathbf{\Phi}_{yu}, \mathbf{\Phi}_{uy}, \mathbf{\Phi}_{uu})$ lying in the affine subspace (10)-(12), the controller $\mathbf{K} = \mathbf{\Phi}_{uy}\mathbf{\Phi}_{yy}^{-1}$ is causal as per (7) and yields the closed-loop responses $(\mathbf{\Phi}_{yy}, \mathbf{\Phi}_{yu}, \mathbf{\Phi}_{uy}, \mathbf{\Phi}_{uu})$.

We are now ready to establish a strongly convex formulation of the optimal control problem under study. Please refer to the Appendix of [19] for a complete proof.

Proposition 2: Consider the LTI system (1). The controller in the form (6) achieving the minimum of the cost functional (3) is given by $\mathbf{K} = \mathbf{\Phi}_{uy}\mathbf{\Phi}_{yy}^{-1}$, where $\mathbf{\Phi}_{uy}, \mathbf{\Phi}_{yy}$ are optimal solutions to the following strongly convex program:

$$\min_{\mathbf{\Phi}} \left\| \begin{bmatrix} \mathbf{L}^{\frac{1}{2}} & 0 \\ 0 & \mathbf{R}^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma}_v^{\frac{1}{2}} & 0 & \mathbf{y}_{x(0)} \\ 0 & \mathbf{\Sigma}_w^{\frac{1}{2}} & 0 \end{bmatrix} \right\|_F^2 \tag{13}$$
$$\text{subject to } (10)-(12)\,,$$

where $\mathbf{y}_{x(0)} = \mathbf{CP}_A(:,0)x(0)$, $\mathbf{L} = \text{blkdiag}(L_0, \cdots, L_{N-1})$, $\mathbf{R} = \text{blkdiag}(R_0, \cdots, R_{N-1})$, $\mathbf{\Sigma}_v = I_N \otimes \Sigma_v$ and $\mathbf{\Sigma}_w = I_N \otimes \Sigma_w$.

When the system parameters (A, B, C, x_0) are known, it is straightforward and efficient to compute the unique globally optimal solution $(\mathbf{\Phi}^\star_{yy}, \mathbf{\Phi}^\star_{yu}, \mathbf{\Phi}^\star_{uy}, \mathbf{\Phi}^\star_{uu})$ of problem (13) with off-the-shelf interior point solvers. The globally optimal control policy is recovered as $\mathbf{K}^\star = \mathbf{\Phi}^\star_{uy}(\mathbf{\Phi}^\star_{yy})^{-1}$. We also remark that, since the noise is Gaussian, the linear policy $\mathbf{u} = \pi^\star(\mathbf{y}) = \mathbf{K}^\star\mathbf{y}$ is optimal with respect to all feedback policies. If the noise is non-Gaussian, $\mathbf{K}^\star$ remains the optimal linear controller, but nonlinear policies may outperform it.

However, it is more challenging to compute $\mathbf{K}^\star$ merely relying on libraries of past input-output trajectories. In the next section, we exploit behavioral theory to provide a non-parametric data-driven version of (13).

IV. Behavioral Input-Output Parametrization

Before moving on, we recall the following definition of persistency of excitation and the result known as the *Fundamental Lemma* for LTI systems [20].

Definition 1: We say that $\mathbf{u}^h_{[0,T-1]}$ is *persistently exciting* (PE) of order L if the Hankel matrix $\mathcal{H}_L(\mathbf{u}^h_{[0,T-1]})$ has full row-rank.

A necessary condition for the matrix $\mathcal{H}_L(\mathbf{u}^h_{[0,T-1]})$ to be full row-rank is that it has at least as many columns as rows. It follows that the trajectory $\mathbf{u}^h_{[0,T-1]}$ must be long enough to satisfy $T \geq (m+1)L - 1$.

Lemma 1 (Theorem 1, [20]): Consider system (1) and assume that (A, B) is controllable and that there is no noise. Let $\{\mathbf{y}^h_{[0,T-1]}, \mathbf{u}^h_{[0,T-1]}\}$ be a historical system trajectory of length T. Then, if $\mathbf{u}_{[0,T-1]}$ is PE of order $n + L$, the signals $\mathbf{y}^\star_{[0,L-1]} \in \mathbb{R}^{pL}$ and $\mathbf{u}^\star_{[0,L-1]} \in \mathbb{R}^{mL}$ are valid trajectories of

[1] Specifically, they have the block lower-triangular sparsities resulting by construction from the expressions (9), the sparsity of $\mathbf{K}$ in (7) and that of $\mathbf{G}$.

(1) if and only if there exists $g \in \mathbb{R}^{T-L+1}$ such that

$$\begin{bmatrix} \mathcal{H}_L(\mathbf{y}^h_{[0,T-1]}) \\ \mathcal{H}_L(\mathbf{u}^h_{[0,T-1]}) \end{bmatrix} g = \begin{bmatrix} \mathbf{y}^\star_{[0,L-1]} \\ \mathbf{u}^\star_{[0,L-1]} \end{bmatrix} . \tag{14}$$

Next, we show how Lemma 1 can be directly exploited to obtain a non-parametric formulation of (13). We work under the following assumptions.

Assumption 1: The data-generating LTI system (1) is such that (A, B) is controllable and (A, C) is observable.

Assumption 2: The following data are available:

i) a *recent* system trajectory of length T_{ini}: $\left\{\mathbf{y}^r_{[0,T_{ini}-1]}, \mathbf{u}^r_{[0,T_{ini}-1]}\right\}$, with $\mathbf{y}^r_{[0,T_{ini}-1]} = \mathbf{y}_{[-T_{ini},-1]}$ and $\mathbf{u}^r_{[0,T_{ini}-1]} = \mathbf{u}_{[-T_{ini},-1]}$,
ii) a *historical* system trajectory of length T: $\left\{\mathbf{y}^h_{[0,T-1]}, \mathbf{u}^h_{[0,T-1]}\right\}$, with $\mathbf{y}^h_{[0,T-1]} = \mathbf{y}_{[-T_h,-T_h+T-1]}$ and $\mathbf{u}^h_{[0,T-1]} = \mathbf{u}_{[-T_h,-T_h+T-1]}$ for $T_h \in \mathbb{N}$ such that $T_h > T + T_{ini}$.

Assumption 3: The historical and recent data are not corrupted by noise.

We will drop Assumption 3 in Section V.

Assumption 4: The historical input trajectory $\mathbf{u}^h_{[0,T-1]}$ is persistently exciting of order $n + T_{ini} + N$, where $T_{ini} \geq l$ and l is the smallest integer such that

$$\begin{bmatrix} C^\mathsf{T} & (CA)^\mathsf{T} & \cdots & (CA^{l-1})^\mathsf{T} \end{bmatrix}^\mathsf{T} ,$$

has full row-rank. Note that if Assumption 1 holds, then $l \leq n$.

A few comments are in order. First, in Assumption 2 the *historical* data are needed to construct a non-parametric system representation, and the *recent* data are exploited to define a cost function that accurately reflects the system initial state $x(0) \in \mathbb{R}^n$. Second, in Assumption 3 we assume that the observed data are noiseless to construct a data-driven optimal control problem that is *equivalent* to (13). We will deal with the noisy case in Section V.

Theorem 1 (Behavioral IOP): Consider the unknown LTI system (1) and let Assumptions 1-4 hold. Let (G, g) be any solutions to the linear system of equations

$$\begin{bmatrix} U_p \\ Y_p \\ U_f \end{bmatrix} \begin{bmatrix} G & g \end{bmatrix} = \begin{bmatrix} 0_{mT_{ini}\times m} & \mathbf{u}^r_{[0,T_{ini}-1]} \\ 0_{pT_{ini}\times m} & \mathbf{y}^r_{[0,T_{ini}-1]} \\ \begin{bmatrix} I_m & 0_{m\times m(N-1)} \end{bmatrix}^\mathsf{T} & 0_{mN\times 1} \end{bmatrix}, \tag{15}$$

where $\begin{bmatrix} U_p \\ U_f \end{bmatrix} = \mathcal{H}_{T_{ini}+N}(\mathbf{u}^h_{[0,T-1]})$ and $\begin{bmatrix} Y_p \\ Y_f \end{bmatrix} = \mathcal{H}_{T_{ini}+N}(\mathbf{y}^h_{[0,T-1]})$. Then, the optimization problem (13) is equivalent to

$$\min_{\mathbf{\Phi}} \left\| \begin{bmatrix} \mathbf{L}^{\frac{1}{2}} & 0 \\ 0 & \mathbf{R}^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma}_v^{\frac{1}{2}} & 0 & Y_f g \\ 0 & \mathbf{\Sigma}_w^{\frac{1}{2}} & 0 \end{bmatrix} \right\|_F^2 \tag{16}$$

$$\text{subject to} \quad \text{IOP}(\mathbf{\Phi}, \text{Toep}(Y_f G)) .$$

Proof: In problem (13), the system parameters $(A, B, C, x(0))$ appear through the terms $\mathbf{G} = \mathbf{C}\mathbf{P}_B$ in the constraints and $\mathbf{C}\mathbf{P}_A x(0)$ in the cost. It is therefore sufficient to show that we are able to substitute both elements with data as per the theorem statement.

Let G be any solution (15). By rearranging the terms, each column of G can be thought as a solution to (14) associated with a zero initial condition and a unitary input $e_i \in \mathbb{R}^m$. Since the hypotheses of Lemma 1 are satisfied for $L = T_{ini} + N$, similar to Proposition 11 of [21] we deduce that $Y_f G$ is the first block-column of the system impulse response matrix, independent of the solution G. Therefore, we can equivalently substitute $\mathbf{G} = \text{Toep}_{p\times m}(Y_f G)$ in the constraints (10)-(11) of problem (13). Finally, note that $Y_f g$ corresponds to the trajectory starting at $x(0)$ (as implicitly defined by the recent trajectory $\mathbf{y}_{[-T_{ini},-1]}$ and $\mathbf{u}_{[-T_{ini},-1]}$) when applying a zero input [21]. Therefore, it corresponds to the true free response starting from $x(0)$. ■

For any solution G of the behavioral impulse response representation (15), the affine constraints (10)-(12) describe all the achievable closed-loop responses for the unknown model and the corresponding controller $\mathbf{K}$. Also, for any solution g of (15), the term $Y_f g$ represents the true free response of the system. As a result, the achieved optimal controller $\mathbf{K}^\star$ and optimal cost $J^\star$ are independent of the chosen solution (G, g) for (15). We have thus characterized a data-driven version of the IOP, based on a preliminary impulse response identification phase enabled by behavioral theory. Theorem 1 further shows that, by exploiting the BIOP, it is straightforward to cast the LQG problem as a strongly convex program.

V. Robust BIOP with Noise-Corrupted Data

The linear system (15) is highly underdetermined when the historical trajectory is very long and noiseless. In particular, any solution (G, g) to (15) gives an exact impulse response matrix and free trajectory of the system. In practice, however, the historical and recent data are corrupted by noise. According to the system equations (1)-(2), we can assume historical and recent trajectories are affected by noise $w^h(t), w^r(t), v^h(t), v^r(t)$ at all time instants, with expected values $\mu^h_w, \mu^r_w, \mu^h_v, \mu^r_v$ and variances $\mathbf{\Sigma}^h_w, \mathbf{\Sigma}^r_w, \mathbf{\Sigma}^h_v, \mathbf{\Sigma}^r_v$ respectively. Hence, the matrix on the left-hand-side of (15) becomes full row-rank almost surely and (15) do not recover the free and impulse responses. This issue is well-known in the behavioral theory literature, and several promising solutions have recently been proposed [8], [9], [12], [14]. We briefly review some of them in the accompanying Arxiv report [19].

Independent of the chosen estimator, we will have that

$$\mathbb{E}[\widehat{\mathbf{G}}] = M_G, \quad \text{Var}(\text{vec}(\widehat{\mathbf{G}})) = \Sigma_G ,$$
$$\mathbb{E}[\widehat{\mathbf{y}}_{\text{free}}] = \mu_y, \quad \text{Var}(\widehat{\mathbf{y}}_{\text{free}}) = \Sigma_y ,$$

where $M_G = \mathbf{G}$ and $\mu_y = \mathbf{y}_{\text{free}}$ if and only if the estimators are unbiased, and where Σ_G, Σ_y are "small" in an appropriate sense. We work under the assumption that, with high-probability, the errors $\left\|\mathbf{G} - \widehat{\mathbf{G}}\right\|$ and $\|\mathbf{y}_{\text{free}} - \widehat{\mathbf{y}}_{\text{free}}\|$ are small; the better the predictor (i.e., smaller bias and variance), the smaller the errors. Motivated as above, we abstract from the particular identification scheme and formalize the following assumption.

Assumption 5: There exist $\epsilon_G > 0$ and $\epsilon_0 > 0$ such that, for any sequence of noisy historical and recent data, with high probability

$$\left\|\mathbf{G} - \widehat{\mathbf{G}}\right\|_2 = \|\boldsymbol{\Delta}\|_2 \leq \epsilon_G, \;\; \|\mathbf{y}_{\text{free}} - \hat{\mathbf{y}}_{\text{free}}\|_2 = \|\boldsymbol{\delta}_0\|_2 \leq \epsilon_0 \,.$$

We denote $\epsilon = \max(\epsilon_G, \epsilon_0)$.

After condensing the effect of noise into a single error parameter $\epsilon > 0$, we are ready to leverage and adapt the analysis technique recently suggested in [5] for infinite-horizon LQG, which follows the philosophy first introduced in [3] for LQR. This allows to quantify the performance degradation with respect to model-based LQG that one incurs when using behavioral models to estimate the system impulse response from noisy data. The first step is to construct a robust version of (16) that is defined in terms of the available noisy historical data. The proof of Proposition 3 is reported in the Appendix of [19]. For simplicity, but without loss of generality, we assume that $\mathbf{L}, \mathbf{R}, \boldsymbol{\Sigma}_w, \boldsymbol{\Sigma}_v$ are identity matrices with appropriate dimensions.

Proposition 3: Assume that historical and recent data are corrupted by noise. Let $\widehat{\mathbf{G}}, \widehat{\mathbf{y}}_{\text{free}}$ be estimators of $\mathbf{G}, \mathbf{y}_{\text{free}}$, respectively, such that Assumption 5 holds with $\epsilon > 0$. Let $J(\mathbf{G}, \mathbf{K}) = \sqrt{\mathbb{E}_{\mathbf{w},\mathbf{v}}\left[\mathbf{y}^\mathsf{T}\mathbf{y} + \mathbf{u}^\mathsf{T}\mathbf{u}\right]}$ denote the square root of the cost in (3). Consider the following model-based worst-case robust optimal control problem:

$$\begin{aligned} \min_{\mathbf{K}} \quad & \max_{\|\boldsymbol{\Delta}\|_2 \leq \epsilon, \; \|\boldsymbol{\delta}_0\|_2 \leq \epsilon} \sqrt{\mathbb{E}_{\mathbf{w},\mathbf{v}}\left[\mathbf{y}^\mathsf{T}\mathbf{y} + \mathbf{u}^\mathsf{T}\mathbf{u}\right]} && (17) \\ \text{subject to} \quad & (4), (5), (6), (7)\,. \end{aligned}$$

Then, problem (17) is equivalent to

$$\min_{\widehat{\boldsymbol{\Phi}}} \max_{\substack{\|\boldsymbol{\Delta}\|_2 \leq \epsilon \\ \|\boldsymbol{\delta}_0\|_2 \leq \epsilon}} \left\| \begin{bmatrix} \widehat{\boldsymbol{\Phi}}_{yy}(I\text{-}\boldsymbol{\Delta}\widehat{\boldsymbol{\Phi}}_{uy})^{-1} & \widehat{\boldsymbol{\Phi}}_{yy}(I\text{-}\boldsymbol{\Delta}\widehat{\boldsymbol{\Phi}}_{uy})^{-1}(\widehat{\mathbf{G}}+\boldsymbol{\Delta}) \\ \widehat{\boldsymbol{\Phi}}_{uy}(I\text{-}\boldsymbol{\Delta}\widehat{\boldsymbol{\Phi}}_{uy})^{-1} & (I\text{-}\widehat{\boldsymbol{\Phi}}_{uy}\boldsymbol{\Delta})^{-1}\widehat{\boldsymbol{\Phi}}_{uu} \end{bmatrix} \times \right.$$

$$\left. \times \begin{bmatrix} I & 0 & \widehat{\mathbf{y}}_{\text{free}} + \boldsymbol{\delta}_0 \\ 0 & I & 0 \end{bmatrix} \right\|_F \tag{18}$$

$$\text{subject to} \quad \text{IOP}(\widehat{\boldsymbol{\Phi}}, \widehat{\mathbf{G}})\,. \tag{19}$$

The robust optimization problem in Proposition 3 is highly non-convex. We therefore proceed with deriving a quasi-convex upper-bound to $J(\mathbf{G}, \mathbf{K})$ to be used for controller synthesis and suboptimality analysis.

A. A tractable robust BIOP formulation

The following lemma serves as the basis to derive a tractable formulation of (18). Its rather lengthy technical proof is reported in the Appendix of [19].

Lemma 2: Let $\epsilon = \max(\epsilon_G, \epsilon_0)$ and assume $\epsilon\left\|\widehat{\boldsymbol{\Phi}}_{uy}\right\|_2 < 1$. Further assume that $\left\|\widehat{\boldsymbol{\Phi}}_{uy}\right\|_2 \leq \alpha$ for $\alpha > 0$. Then, we have

$$J(\mathbf{G}, \mathbf{K}) \leq \frac{1}{1 - \epsilon\left\|\widehat{\boldsymbol{\Phi}}_{uy}\right\|_2} \times \tag{20}$$

$$\times \left\| \begin{bmatrix} \sqrt{1+h(\epsilon, \alpha, \widehat{\mathbf{G}})+h(\epsilon, \alpha, \widehat{\mathbf{y}}_{\text{free}})}\,\widehat{\boldsymbol{\Phi}}_{yy} & \widehat{\boldsymbol{\Phi}}_{yu} & \widehat{\boldsymbol{\Phi}}_{yy}\widehat{\mathbf{y}}_{\text{free}} \\ \sqrt{1 + h(\epsilon, \alpha, \widehat{\mathbf{y}}_{\text{free}})}\,\widehat{\boldsymbol{\Phi}}_{uy} & \widehat{\boldsymbol{\Phi}}_{uu} & \widehat{\boldsymbol{\Phi}}_{uy}\widehat{\mathbf{y}}_{\text{free}} \end{bmatrix} \right\|_F,$$

where $h(\epsilon, \alpha, \mathbf{Y}) = \epsilon^2(2+\alpha\|\mathbf{Y}\|_2)^2 + 2\epsilon\,\|\mathbf{Y}\|_2\,(2+\alpha\,\|\mathbf{Y}\|_2)$.

Exploiting the reformulation idea first introduced in [22] and utilized for analysis in [5], we are now ready to establish a quasi-convex reformulation of problem (18).

Theorem 2: Given estimation errors ϵ_G, ϵ_0 with $\epsilon = \max(\epsilon_G, \epsilon_0)$, and for any $\alpha > 0$, the minimal cost of problem (17) is upper bounded by the minimal cost of the following quasi-convex program:

$$\begin{aligned} \min_{\gamma \in [0, \epsilon^{-1})} \frac{1}{1 - \epsilon\gamma} \min_{\widehat{\boldsymbol{\Phi}}} \quad & J_{inner} && (21) \\ \text{subject to} \quad & \text{IOP}(\widehat{\boldsymbol{\Phi}}, \widehat{\mathbf{G}}), \quad \left\|\widehat{\boldsymbol{\Phi}}_{uy}\right\|_2 \leq \min(\gamma, \alpha)\,. \end{aligned}$$

where J_{inner} is equal to

$$\left\| \begin{bmatrix} \sqrt{1+h(\epsilon, \alpha, \widehat{\mathbf{G}})+h(\epsilon, \alpha, \widehat{\mathbf{y}}_{\text{free}})}\,\widehat{\boldsymbol{\Phi}}_{yy} & \widehat{\boldsymbol{\Phi}}_{yu} & \widehat{\boldsymbol{\Phi}}_{yy}\widehat{\mathbf{y}}_{\text{free}} \\ \sqrt{1 + h(\epsilon, \alpha, \widehat{\mathbf{y}}_{\text{free}})}\,\widehat{\boldsymbol{\Phi}}_{uy} & \widehat{\boldsymbol{\Phi}}_{uu} & \widehat{\boldsymbol{\Phi}}_{uy}\widehat{\mathbf{y}}_{\text{free}} \end{bmatrix} \right\|_F.$$

Proof: Directly follows from Lemma 2 and [5, Theorem 3.2]. ■

First, notice that the inner minimization problem in (21) is strongly convex for a fixed γ, and that the outer function $(1-\epsilon\gamma)^{-1}$ is monotonically increasing in γ. Hence, it is well-known that the overall program can be efficiently solved by golden search on γ and solving the corresponding instances of the inner program. Second, we explicitly take into account the effect of an unknown and noisy initial state $x(0) \in \mathbb{R}^n$ through the parameter $\widehat{\mathbf{y}}_{\text{free}}$. Assuming $x(0) = 0$ as per [15] may not be realistic for practical purposes, as the user initially lets the system free to evolve in order to harvest data. Furthermore, the following analysis will show that, for finite-horizon control problems, the suboptimality strongly depends on $x(0) \in \mathbb{R}^n$ as a function of $\|\mathbf{y}_{\text{free}}\|_2^2$. Last, we note that the constraint on $\|\widehat{\boldsymbol{\Phi}}_{uy}\|_2$ is the main source of suboptimality with respect to the true LQG problem (13); as pointed out in [3], [5], [15], this additional constraint enforces stronger disturbance rejection properties, for which we have to pay in terms of performance. We are now ready to quantify the suboptimality of (21) with respect to (13).

VI. Suboptimality Analysis

In this section, we denote as $\mathbf{K}^\star, \boldsymbol{\Phi}^\star$ the optimal controller and corresponding closed-loop responses for the real problem (13). We denote as $\widehat{\mathbf{K}}^\star, \widehat{\boldsymbol{\Phi}}^\star$ the optimal controller and corresponding closed-loop responses for the quasi-convex program (21) and let $J^\star = J(\mathbf{G}, \mathbf{K}^\star)$ and $\hat{J} = J(\mathbf{G}, \widehat{\mathbf{K}}^\star)$.

Next, inspired by the analysis in [5], we show that if ϵ is small enough it holds

$$\frac{\hat{J}^2 - J^{\star 2}}{J^{\star 2}} = \mathcal{O}\left(\epsilon\right)\,.$$

In other words, for a small estimation error ϵ on the impulse response, applying controller $\widehat{\mathbf{K}}^\star$ (which is solely computed with noisy data) to the *real* plant achieves almost optimal closed-loop performance. We start with a lemma that analytically characterizes a feasible solution to problem (21), whose suboptimality is exploited in further characterizing the suboptimality bound. The proofs of Lemma 3 and Theorem 3 are reported in the Appendix of [19].

Lemma 3 (Feasible solution): Let $\eta = \epsilon \left\|\mathbf{\Phi}^\star_{uy}\right\|_2$, and select $\alpha \geq \sqrt{2}\frac{\eta}{\epsilon(1-\eta)}$. Then, if $\eta < \frac{1}{5}$, the following expressions

$$\begin{aligned}
&\widetilde{\mathbf{\Phi}}_{yy} = \mathbf{\Phi}^\star_{yy}(I+\mathbf{\Delta}\mathbf{\Phi}^\star_{uy})^{-1},\ \widetilde{\mathbf{\Phi}}_{yu} = \mathbf{\Phi}^\star_{yy}(I+\mathbf{\Delta}\mathbf{\Phi}^\star_{uy})^{-1}(\mathbf{G}-\mathbf{\Delta}),\\
&\widetilde{\mathbf{\Phi}}_{uy} = \mathbf{\Phi}^\star_{uy}(I+\mathbf{\Delta}\mathbf{\Phi}^\star_{uy})^{-1},\ \widetilde{\mathbf{\Phi}}_{uu} = (I+\mathbf{\Phi}^\star_{uy}\mathbf{\Delta})^{-1}\mathbf{\Phi}^\star_{uu},\\
&\widetilde{\gamma} = \frac{\sqrt{2}\eta}{\epsilon(1-\eta)},
\end{aligned} \tag{22}$$

provide a feasible solution to problem (21).

Theorem 3: Suppose that $\frac{5\sqrt{2}}{4}\left\|\mathbf{\Phi}^\star_{uy}\right\|_2 \leq \alpha \leq 5\left\|\mathbf{\Phi}^\star_{uy}\right\|_2$ and that $\epsilon < \frac{1}{5\left\|\mathbf{\Phi}^\star_{uy}\right\|_2}$. Then, when applying the optimal solution $\widehat{\mathbf{K}}^\star$ of (21) to the true plant $\mathbf{G}$, the relative error with respect to the true optimal cost is upper bounded as

$$\begin{aligned}
\frac{\hat{J}^2 - J^{\star 2}}{J^{\star 2}} &\leq 20\epsilon\left\|\mathbf{\Phi}^\star_{uy}\right\|_2 + 4(M+V)\\
&= \mathcal{O}\left(\epsilon\left\|\mathbf{\Phi}^\star_{uy}\right\|_2(\|\mathbf{G}\|_2^2 + \|\mathbf{y}_{\text{free}}\|_2^2)\right),
\end{aligned}$$

where

$$\begin{aligned}
&M = h(\epsilon,\alpha,\widehat{\mathbf{G}}) + h(\epsilon,\alpha,\widehat{\mathbf{y}}_{\text{free}}) + h(\epsilon,\left\|\mathbf{\Phi}^\star_{uy}\right\|_2,\mathbf{G})\\
&\quad + h(\epsilon,\left\|\mathbf{\Phi}^\star_{uy}\right\|_2,\mathbf{y}_{\text{free}}),\\
&V = h(\epsilon,\alpha,\widehat{\mathbf{y}}_{\text{free}}) + h(\epsilon,\left\|\mathbf{\Phi}^\star_{uy}\right\|_2,\mathbf{y}_{\text{free}}),
\end{aligned}$$

and $h(a,b,\mathbf{Y}) = a^2(2+b\|\mathbf{Y}\|_2)^2 + 2a\|\mathbf{Y}\|_2(2+b\|\mathbf{Y}\|_2)$.

Theorem 3 shows that the relative performance of the robust BIOP formulation (21) with respect to its exact non-noisy version (16) decreases linearly with ϵ, as long as ϵ is small enough to guarantee $\epsilon\left\|\mathbf{\Phi}^\star_{uy}\right\|_2 < \frac{1}{5}$. The bound also grows quadratically with the norm of the true impulse and free responses, which implies that an unstable system will be difficult to control for a long horizon. Note that it is appropriate to choose α not too large, and specifically $\alpha \leq 5\left\|\mathbf{\Phi}_{uy}\right\|_2 < \epsilon^{-1}$ in order for the scaling of $h(\epsilon,\alpha,\widehat{\mathbf{G}})$ in terms of ϵ not to dominate over $h(\epsilon,\left\|\mathbf{\Phi}^\star_{uy}\right\|_2,\mathbf{G})$. Our rate in terms of ϵ matches that of [3], [5], which are valid in infinite-horizon. In spite of the additional challenges of considering a noisy unknown initial state $x(0) \in \mathbb{R}^n$ and noisy output-feedback, our rate also matches the one achieved with the approach of [15] valid for $x(0) = 0$ and state-feedback.

VII. Conclusions

We have proposed the BIOP, a method for the design of optimal output-feedback controllers which directly embeds historical input-output trajectories in its formulation. When these historical data are noiseless, the BIOP is equivalent to the standard IOP and recovers an optimal LQG controller. In the presence of noise-corrupted data, we propose a robust version of the BIOP that explicitly incorporates the estimated uncertainty level and that can be solved efficiently through convex programming. By exploiting recently developed analysis techniques, the suboptimality of the obtained solution is quantified and compared with the nominal LQG solution. Furthermore, the developed framework is readily compatible with state-of-the-art behavioral estimation and prediction techniques, e.g. [12], [14].

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