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Short Communication

A comment on performance guarantees of a greedy algorithm for minimizing a supermodular set function on comatroid

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ABSTRACT

We provide a counterexample to the performance guarantee obtained in the paper "Il'ev, V., Linker, N., 2006. Performance guarantees of a greedy algorithm for minimizing a supermodular set function on comatroid", which was published in Volume 171 of the European Journal of Operational Research. We point out where this error originates from in the proof of the main theorem.

1. Problem formulation

Let U be a finite set with $|U| = n$. Let hereditary system $\mathcal{H} = (U, \mathcal{D})$ denote a comatroid, where the family $\mathcal{D}$ denotes the dependence system, and let $f : 2^U \to \mathbb{R}_+$ denote a set function. Il'ev and Linker (2006) consider solving

$$\min\{f(X) : X \in \mathcal{C}\}, \tag{1}$$

where $\mathcal{C}$ is the family given by all circuits of a comatroid $\mathcal{H} = (U, \mathcal{D})$ with girth p, and f is supermodular, nonincreasing, and $f(U) = 0$. We refer to (Il'ev & Linker, 2006) and (Il'ev, 2003) for the definitions relating to comatroids and hereditary systems.

This problem is known to be NP-hard since the well-known p-median problem can be captured as a special case. As a heuristic, Il'ev and Linker (2006) propose the greedy descent algorithm (also known as reverse greedy or stingy algorithm), which proceeds as follows:

Greedy descent algorithm:

Step 0: Set $X_0 = U$. Go to Step 1.
Step i: ($i \geq 1$): Select $x_i \in X_{i-1}$ such that

$$d_{x_i}(X_{i-1}) = \min_{\substack{x \in X_{i-1} \\ X_{i-1}\setminus\{x\} \in \mathcal{D}}} d_x(X_{i-1}), \tag{2}$$

where $d_x(X) = f(X \setminus \{x\}) - f(X)$ is the marginal increment in f when removing $\{x\}$ from the set X. Set $X_i = X_{i-1} \setminus \{x_i\}$. If $i = n - p$, then stop. Otherwise go to Step $i + 1$.
End.

The paper contained the following theorem regarding the suboptimality bound of the greedy heuristic when applied to (1).

Theorem 1 (Il'ev and Linker, 2006, Theorem 1)**.** *Let OPT be an optimal solution to (1) on an arbitrary comatroid and GR be the solution returned by the greedy descent algorithm. Then,*

$$\frac{f(\text{GR})}{f(\text{OPT})} \leq \frac{1}{t}\left(\left(1 + \frac{t}{q}\right)^q - 1\right),$$

where $q = n - p$, $t = s/1 - s$, and s is the solution to the following problem

$$s = \max_{\substack{x \in U, \\ f(\{x\}) < f(\emptyset)}} \frac{(f(\emptyset) - f(\{x\})) - (f(U \setminus \{x\}) - f(U))}{(f(\emptyset) - f(\{x\}))}.$$

In the following, we provide a counterexample showing that the guarantee in Theorem 1 does not necessarily hold. Then, we point out the mistake found in (Il'ev and Linker, 2006, Lemma 1), which is utilized in constructing the linear program (Il'ev and Linker, 2006, equation (8)) in the proof of Theorem 1.

2. Counterexample

Set $n = 4$, $U = \{1, 2, 3, 4\}$. Consider the following nonincreasing supermodular function:

$$f(\emptyset) = 6,\ f(\{1\}) = 4,\ f(\{2\}) = 5,\ f(\{3\}) = 4,\ f(\{4\}) = 4,$$
$$f(\{3,4\}) = 2,\ f(\{1,2\}) = 3,\ f(\{2,4\}) = 3,$$
$$f(\{2,3\}) = 3,\ f(\{1,4\}) = 2,\ f(\{1,3\}) = 2,$$
$$f(\{1,2,3\}) = 1,\ f(\{2,3,4\}) = 1,\ f(\{1,3,4\}) = 1,$$
$$f(\{1,2,4\}) = 1, f(\{1,2,3,4\}) = f(U) = 0.$$

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Compute the steepness s of function f:

$$s = \frac{(6-4)-(1-0)}{(6-4)} = 0.5.$$

Hence, $t = 1$. Define the comatroid $\mathcal{H} = (U, \mathcal{D})$ as follows:

$$\mathcal{D} = \{U = \{1,2,3,4\}, \{2,3,4\}, \{1,3,4\}, \{1,2,4\}, \{1,2,3\}, \{1,2\}, \{1,4\}, \{2,3\}, \{3,4\}\}.$$

Note that girth is given by $p = 2$, thus $q = 2$. This comatroid was previously studied in (Il'ev and Linker, 2006, Remark 3). Clearly, the family given by all circuits of this comatroid is $\mathcal{C} = \{\{1,2\}, \{1,4\}, \{2,3\}, \{3,4\}\}$.

Consider (1) with the comatroid $\mathcal{H} = (U, \mathcal{D})$ and the objective f. The greedy descent algorithm can find the solution $\mathrm{GR} = \{1,2\}$, whereas an optimal solution is given by $\mathrm{OPT} = \{3,4\}$. Theorem 1 claims

$$\frac{3}{2} = \frac{f(\mathrm{GR})}{f(\mathrm{OPT})} \le \frac{1}{t}\left(\left(1+\frac{t}{q}\right)^q - 1\right) = \frac{1}{1}\left(\left(1+\frac{1}{2}\right)^2 - 1\right) = 1.25,$$

which is not correct. As a remark, observe that in this example the selection done in Step 1 by (2) is not unique. We can obtain $\mathrm{GR} = \{1,2\}$ (or $\mathrm{GR} = \{2,3\}$) if $X_1 = \{1,2,3\}$ is chosen in Step 1. If $X_1 = \{1,3,4\}$, greedy descent can find an optimal solution. This statement also holds for $X_1 = \{2,3,4\}$ and $X_1 = \{1,2,4\}$.

3. The error in the proof of the main theorem

Denote the complements by $\overline{X} = U \setminus X$. The proof of (Il'ev and Linker, 2006, Theorem 1) relies on (Il'ev and Linker, 2006, Lemma 1). This lemma exploits the following inequality in its proof.

Inequality 1.

$$\sum_{b \in \overline{\mathrm{OPT}} \setminus \overline{X}_{i-1}} d_b(X_{i-1}) \ge |\overline{\mathrm{OPT}} \setminus \overline{X}_{i-1}| d_{x_i}(X_{i-1}).$$

The above inequality and the resulting (Il'ev and Linker, 2006, Lemma 1) is then utilized in constructing the linear program in (Il'ev and Linker, 2006, Eq. (8)). However, Inequality 1 is not necessarily true. For instance, in the above counterexample, $x_2 = 3$, $X_1 = \{1,2,3\}$, $\mathrm{OPT} = \{3,4\}$, and $\overline{\mathrm{OPT}} \setminus \overline{X}_1 = \{1,2\}$. For $i = 2$, inserting $d_1(X_1) = 2$, $d_2(X_1) = 1$, and $d_3(X_1) = 2$ into Inequality 1, we obtain

$$d_1(X_1) + d_2(X_1) = 2 + 1 \ge 2 \times 2 = 2 \times d_3(X_1),$$

which is not correct.

As an insight, the authors conclude Inequality 1 using the following statement:

"By (2), for every $b \in \overline{\mathrm{OPT}} \setminus \overline{X}_{i-1}$, it holds that $d_b(X_{i-1}) \ge d_{x_i}(X_{i-1})$."

This is equivalent to

$$d_b(X_{i-1}) \ge d_{x_i}(X_{i-1}), \ \forall b \in X_{i-1} \setminus \mathrm{OPT}.$$

For the above inequality to hold by (2), for any $b \in X_{i-1} \setminus \mathrm{OPT}$, we should have $X_{i-1} \setminus b \in \mathcal{D}$. This is not necessarily true. For instance, in the above example, $X_1 = \{1,2,3\}$ and $\mathrm{OPT} = \{3,4\}$, but $2 \in X_{i-1} \setminus \mathrm{OPT}$ is not considered by the step found in (2), since $X_1 \setminus \{2\} \notin \mathcal{D}$. As a remark, the above inequality is correct for p-uniform comatroids, that is, $\mathcal{D} = \{D \subset U : |D| \ge p\}$. Note that from the point of view of applications this is an important case because the well-known p-median problem is a particular case of the considered problem just on p-uniform comatroids.

4. Correction to the error in Inequality 1

In our works (Guo, Karaca, Summers, & Kamgarpour, 2019a; 2019b), we studied a greedy heuristic for a problem similar to (1) where the constraint set is instead the base of a matroid, and the objective is neither supermodular nor submodular but characterized by the notions of curvature and submodularity ratio. Invoking ideas from (Guo, Karaca, Summers, and Kamgarpour, 2019b, Lemma 4), it is possible to revise and correct Inequality 1.

Inequality 2.

$$\sum_{b \in \overline{\mathrm{OPT}} \setminus \overline{X}_{i-1}} d_b(X_{i-1}) \ge (q-(i-1)) d_{x_i}(X_{i-1}).$$

Proof. From the properties of comatroids derived originally in (Il'ev, 2003, Theorem 2: Statement (D2)), it can be verified that there exist $|X_{i-1}| - |\mathrm{OPT}|$ distinct elements from $X_{i-1} \setminus \mathrm{OPT}$ such that after the exclusion of these elements from X_{i-1}, we still obtain a set that lies in the comatroid. Let $R \subseteq X_{i-1} \setminus \mathrm{OPT}$ denote one such subset with exactly $|X_{i-1}| - |\mathrm{OPT}| = n - (i-1) - p = q - (i-1)$ elements. We then obtain the following,

$$\sum_{b \in \overline{\mathrm{OPT}} \setminus \overline{X}_{i-1}} d_b(X_{i-1}) = \sum_{b \in X_{i-1} \setminus \mathrm{OPT}} d_b(X_{i-1}) \ge \sum_{b \in R} d_b(X_{i-1}) \ge (q-(i-1)) d_{x_i}(X_{i-1}).$$

The first equality comes from $\overline{\mathrm{OPT}} \setminus \overline{X}_{i-1} = X_{i-1} \setminus \mathrm{OPT}$. The first inequality follows since function d maps to nonnegative real numbers. The fact that $X_{i-1} \setminus \{x\} \in \mathcal{D}$ for all $x \in R$ and the definition in (2) conclude the last inequality. □

In contrast to Inequality 1, Inequality 2 involves a factor that is independent of the greedy and the optimal solutions. However, this factor is smaller since $(q-(i-1)) \le |\overline{\mathrm{OPT}} \setminus \overline{X}_{i-1}|$. Revisiting our counterexample, for $i = 2$ inserting $d_1(X_1) = 2$, $d_2(X_1) = 1$, and $d_3(X_1) = 2$ into Inequality 2, we obtain

$$d_1(X_1) + d_2(X_1) = 2 + 1 \ge 1 \times 2 = 1 \times d_3(X_1),$$

which is correct. As a future work, it would be interesting to refine (Il'ev and Linker, 2006, Theorem 1) according to Inequality 2. We would like to point out that we believe (Il'ev and Linker, 2006, Theorem 2) is correct for general comatroids. (Il'ev and Linker, 2006, Theorem 3) is also correct for general comatroids, since this theorem can be proved by another method similar to the method of proving (Il'ev, 2003, Theorem 5). Finally, (Il'ev & Il'eva, 2018) provides another bound on the worst-case behaviour of the reverse greedy algorithm for the considered problem. This bound $(1+t)$ is often better than the bound in (Il'ev and Linker, 2006, Theorem 1), see (Il'ev and Il'eva, 2018, Remark 4).

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